A Study of Rationality and Evolution in Games of Bargaining Under Threat

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Ingredients of Game Theory

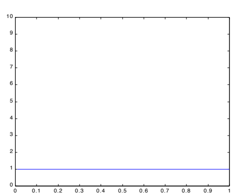
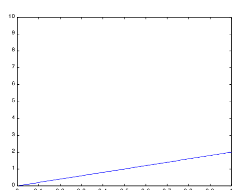
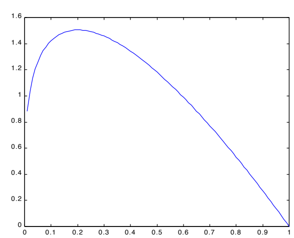
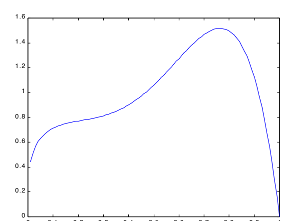
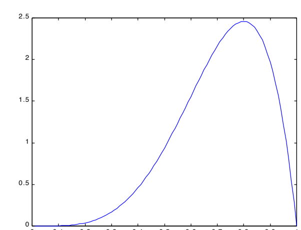
- Competitive situations with at least 2 players who have decisions to make (strategies) with preferences for outcomes
- An Example: The Duopoly
  - Two firms with similar products (Coke v. Pepsi) compete over business
  - Firms choose price
  - Quantity determined by demand; Yields profit: P*Q
  - Firms prefer higher profit
- Solution Concept – Nash Equilibrium
  - A strategy for each player; given what the other players are playing, no other strategy will yield a higher payoff
  - These exist in single shot games

Evolutionary Game Theory

- LARGE population of players each programmed to a strategy
- Players drawn from big pot randomly to play against each other
- Replicator Dynamics
  - A selection mechanism
  - Darwinian "Survival of the Fittest"
  - Individual fitness is measured by payoffs
  - Pure strategies are copied from parent to child flawlessly

How Populations are Described

- Given: Initial Mix of the Population
  - For continuous case – Density function *μ*
    - [0,1] = space of strategies
    - $\mu > 0, \int_0^1 \mu(x)dx = 1$
  - $\int_a^b \mu(x)dx$ is the probability *x* in [a,b] is played
- Examples:

Payoff Structures

- Payoff against another player: P(x,y), Q(x,y)
- Payoff against a population:

$$P^*(x|\lambda_t) = \int_0^1 P(x,y)\lambda_t(y)dy$$

- Fitness of a population

$$F(\mu_t|\lambda_t) = \int_0^1 P^*(x|\lambda_t)\mu_t(x)dx$$

The Replicator Dynamics

- The Single-Population Replicator Dynamics

$$\frac{d\mu_t(x)}{dt} = \mu_t(x)\left(P^*(x|\mu_t) - F(\mu_t)\right)$$

- The Two-Population Replicator Dynamics:

$$\frac{\partial \mu_t(x)}{\partial t} = \mu_t(x)(P^*(x|\lambda_t) - F(\mu_t|\lambda_t))$$

$$\frac{\partial \lambda_t(y)}{\partial t} = \lambda_t(y)(Q^*(y|\mu_t) - G(\lambda_t|\mu_t))$$

Existing Results

- Given a single population with concave payoff function
- Given conditions that ensure dynamics remain in the interior (0,1)
- Under the replicator dynamics, the population will converge to the Dirac mass at the unique interior Nash equilibrium

Extending Results to Two Populations

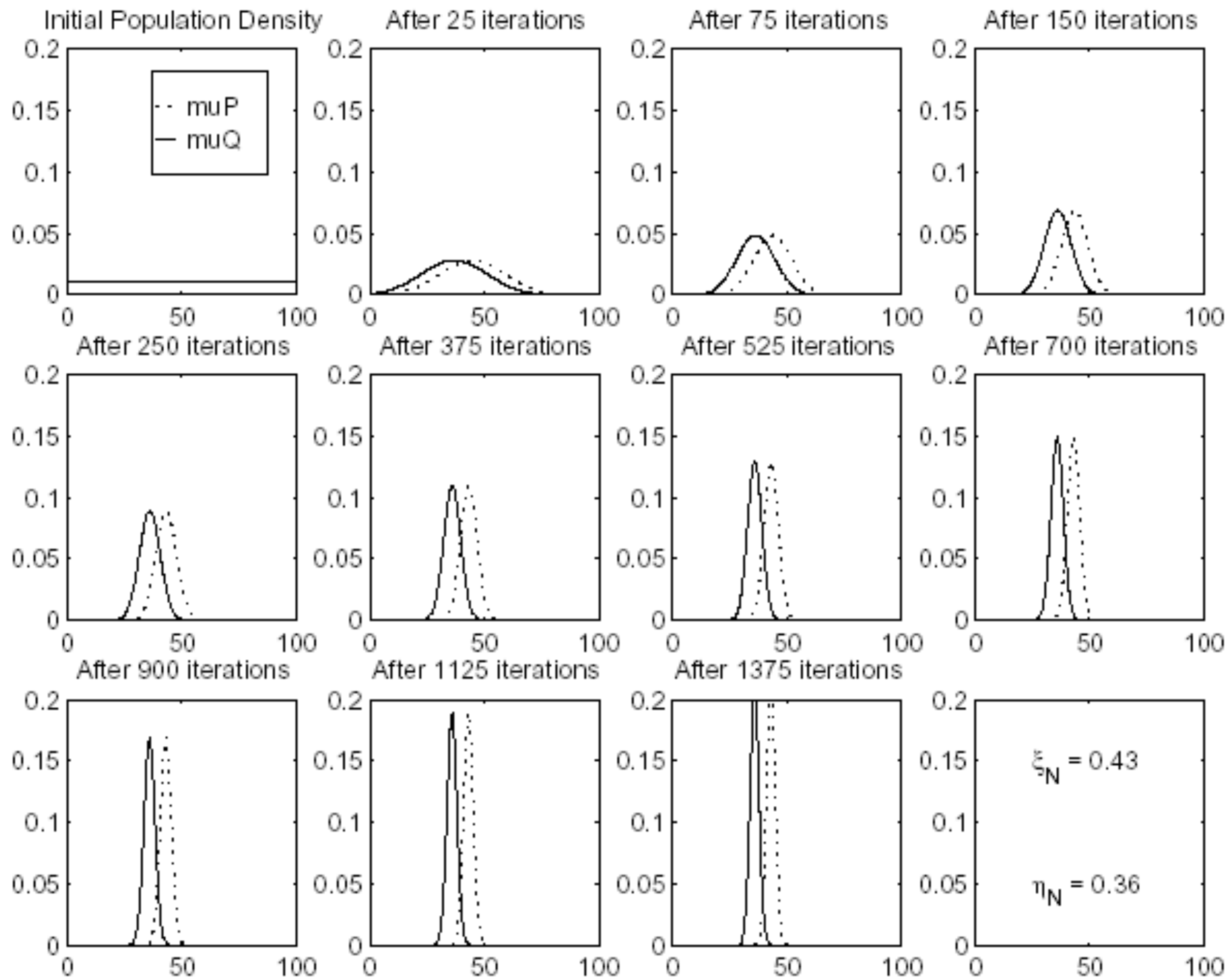
- Assumptions
  - Payoff functions that are concave in decision maker’s variable
  - Boundary conditions to ensure dynamics remains in (0,1) X (0,1)
- The populations will each concentrate their mass to a unique interior peak

Experimental Results I

- P(x,y) = x(1-2x+2y)
- Q(x,y) = y(1-2y+x)
- Uniform Initial Density
- Computed Nash Equilibrium:

$\xi_N = 0.4286$

$\eta_N = 0.3571$

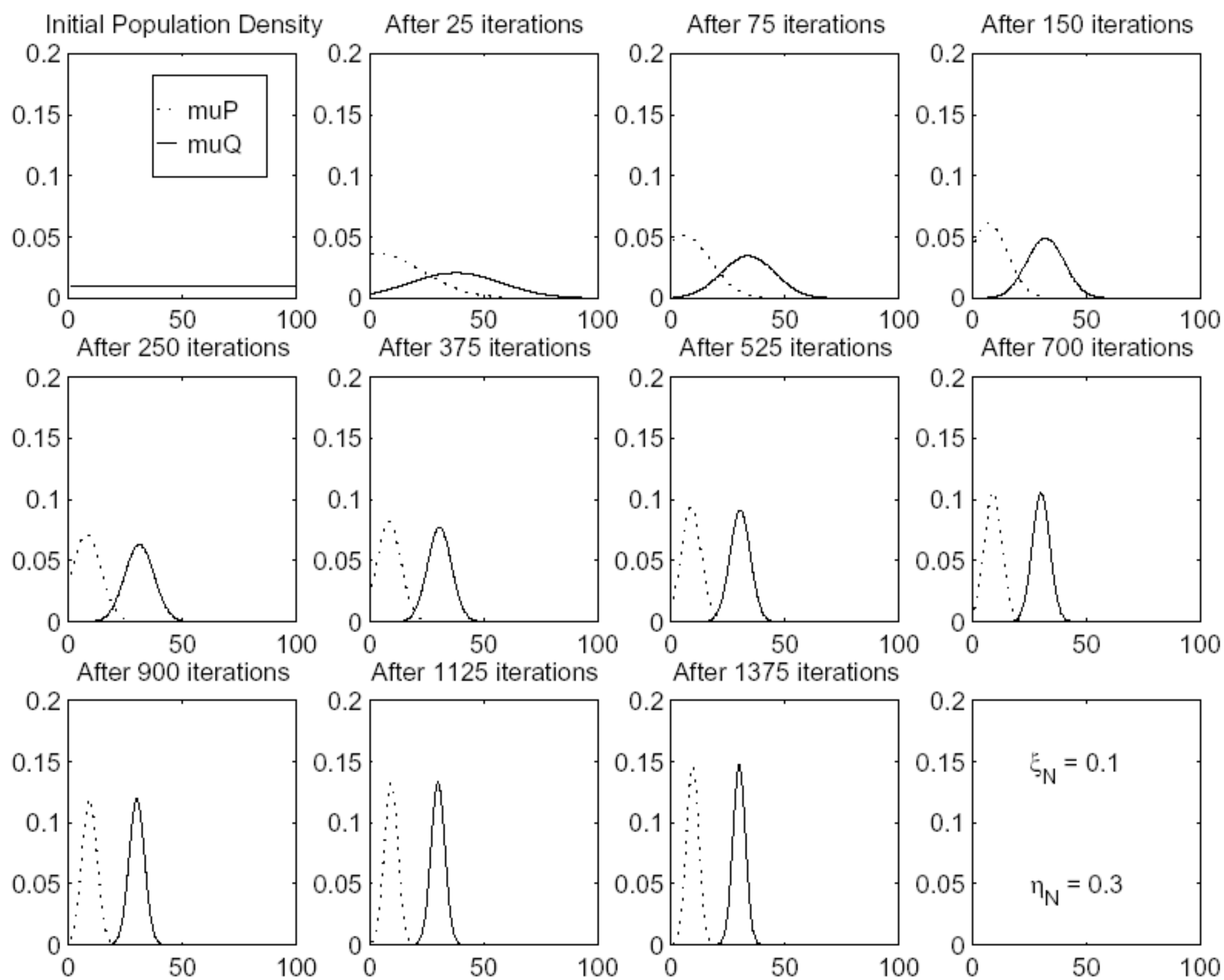


Experimental Results II

- P(x,y) = x(0.5-x-y)
- Q(x,y) = y(0.5-y+x)
- Uniform Initial Density
- Computed Nash Equilibrium:

$\xi_N = 0.1$

$\eta_N = 0.3$



Extending Results…

- Let ξ(t) and η(t) be the unique peaks of $\mu_t(x)$ and $\lambda_t(y)$ respectively
- Then Peak Behavior is described by:

$$\frac{d\xi(t)}{dt} = \frac{-\frac{\partial P^*}{\partial x}(\xi|\lambda_t)}{\left(\frac{\partial}{\partial x}\left(\frac{1}{\mu_0}\frac{\partial \mu_0}{\partial x}\right) + \int_0^t \frac{\partial^2 P^*}{\partial x^2}(x|\lambda_\tau)d\tau\right)_{x=\xi}}$$

$$\frac{d\eta(t)}{dt} = \frac{-\frac{\partial Q^*}{\partial y}(\eta|\mu_t)}{\left(\frac{\partial}{\partial y}\left(\frac{1}{\lambda_0}\frac{\partial \lambda_0}{\partial y}\right) + \int_0^t \frac{\partial^2 Q^*}{\partial y^2}(y|\mu_\tau)d\tau\right)_{y=\eta}}$$

The Linear Quadratic Case

- Payoff for $\mu_t(x)$: $P(x,y) = x(a-bx+cy)$
- Payoff for $\lambda_t(y)$: $Q(x,y) = y(d-ey+fx)$

$$\frac{d\xi(t)}{dt} = -\frac{a - 2b\xi + cEY}{K_\mu(\xi) - 2bt} \qquad \frac{d\eta(t)}{dt} = -\frac{d - 2e\eta + fEX}{K_\lambda(\eta) - 2et}$$

- EY = Expectation of λ; EX = Expectation of μ
  - EY-> η, EX -> ξ as t increases
- $K_\mu(\xi)$, $K_\lambda(\eta)$ are functions of ξ, η respectively
  - $(1/t)K_\mu(\xi)$, $(1/t)K_\lambda(\eta) \to 0$ as t increases

- Change variable to Nash equilibrium

$$\xi_N = \frac{cd+2ea}{4eb-cf} \quad \eta_N = \frac{2bd+af}{4eb-cf} \qquad \xi' = \xi - \xi_N \text{ and } \eta' = \eta - \eta_N$$

$$\frac{\partial \xi'}{\partial t} = \frac{1}{t} \cdot \frac{2b\xi' - c(\eta' + \phi_y(t))}{\frac{1}{t}K_\mu(\eta' + \eta_N) - 2b} \quad \text{and} \quad \frac{\partial \eta'}{\partial t} = \frac{1}{t} \cdot \frac{2e\eta' - f(\xi' + \phi_x(t))}{\frac{1}{t}K_\lambda(\xi' + \xi_N) - 2e}$$

- Change variable: s = ln t

$$\frac{\partial \xi'}{\partial s} = \frac{2b\xi' - c(\eta' + \phi_y(e^s))}{e^{-s}K_\mu(\eta' + \eta_N) - 2b} \quad \text{and} \quad \frac{\partial \eta'}{\partial s} = \frac{2e\eta' - f(\xi' + \phi_x(e^s))}{e^{-s}K_\lambda(\xi' + \xi_N) - 2e}$$

- Reduces to linear system in the limit: $\frac{dx}{ds} = Ax$

$$A = \begin{pmatrix} -1 & \frac{c}{2b} \\ \frac{f}{2e} & -1 \end{pmatrix}, x = \begin{pmatrix} \xi' \\ \eta' \end{pmatrix}$$

- We can write our system in the form $\frac{dx}{ds} = Ax + G(x,s)$ where

$$G(x,s) = \begin{pmatrix} \frac{2b\xi' - c\eta'}{e^{-s}K_\mu(\eta'+\eta_N)-2b} \\ \frac{2e\eta' - f\xi'}{e^{-s}K_\lambda(\xi'+\xi_N)-2e}. \end{pmatrix} - Ax$$

- G(x,s) converges to 0
- When the eigenvalues of A have negative real parts, we can conclude that the solutions to our system are asymptotically stable.
- Namely, the peak will converge to the Nash equilibrium